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A COUNTEREXAMPLE FOR THE TROTTER PRODUCT FORMULA. (U)

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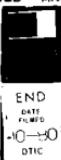
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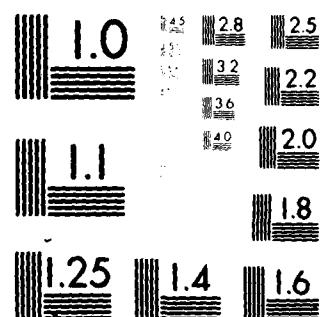
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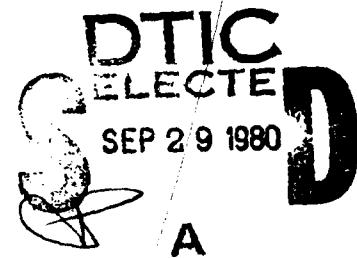
A COUNTEREXAMPLE FOR THE TROTTER  
PRODUCT FORMULA

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A COUNTEREXAMPLE FOR THE TROTTER PRODUCT FORMULA.

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Thomas G. Kurtz and Michel Pierre

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ABSTRACT

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We exhibit here two linear  $m$ -accretive operators  $A_1$  and  $A_2$

whose sum is  $m$ -accretive but for which the associated product formulas

$\left[s^{A_1}(\frac{t}{n})s^{A_2}(\frac{t}{n})\right]^n$  and  $\left[(I + \frac{t}{n}A_1)^{-1}(I + \frac{t}{n}A_2)^{-1}\right]^n$  do not converge.

AMS (MOS) Subject Classifications: Primary 47H15; Secondary 34G05, 35K55

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## SIGNIFICANCE AND EXPLANATION

A wide variety of partial differential equations as well as other equations can be written as ordinary differential equations of the form  $u'(t) + Au(t) = 0$ , where  $u$  takes values in a linear space  $X$  and  $A$  is an operator on  $X$ . The solution is given by  $u(t) = S(t)u(0)$  where  $S(t)$  is a semigroup of operators on  $X$ . In many cases the operator  $A$  can be written as the sum  $A_1 + A_2$  of (possibly simpler) operators where  $A_1$  and  $A_2$  correspond to semigroups  $S_1(t)$  and  $S_2(t)$ . Under appropriate conditions, the Trotter product formula  $S(t)f = \lim_{n \rightarrow \infty} \left[ S_1\left(\frac{t}{n}\right)S_2\left(\frac{t}{n}\right) \right]^n f$  relates  $S(t)$  to  $S_1(t)$  and  $S_2(t)$  and provides one approach to the study of  $S(t)$ .

While various sufficient conditions for the validity of this limit are known, no satisfactory necessary conditions are known even when  $A_1$  and  $A_2$  are linear.

As part of the effort to understand the limitations on the validity of the product formula, we give an example in which  $A_1$ ,  $A_2$  and  $A_1 + A_2$  are all m-accretive but the corresponding semigroups do not satisfy the

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A COUNTEREXAMPLE FOR THE TROTTER PRODUCT FORMULA

Thomas G. Kurtz<sup>1</sup> and Michel Pierre<sup>1,2</sup>

In [10], Trotter proved the following result: given  $-A_1$ ,  $-A_2$  the infinitesimal generators of two strongly continuous semigroups  $S_1(t)$ ,  $S_2(t)$  of linear contractions on a Banach space  $X$ , if  $-(\overline{A_1 + A_2})$  (the closure of  $-(A_1 + A_2)$ ) is also the generator of such a semigroup, say  $S_3(t)$ , then, for any  $f \in X$ :

$$(1) \quad \forall t \in [0, \infty), \lim_{n \rightarrow \infty} \left[ S_1\left(\frac{t}{n}\right) S_2\left(\frac{t}{n}\right) \right]^n f = S_3(t)f.$$

Many attempts arose in the literature to extend this result to the case of nonlinear semigroups of contractions. In this context a natural question is: given  $A_1$ ,  $A_2$  two m-accretive operators on  $X$  such that  $A_3 = A_1 + A_2$  is also m-accretive, is (1) true for the semigroups of contractions "generated" (in the sense of Crandall-Liggett [5]) by  $-A_1$ ,  $-A_2$  and  $-A_3$  and for any  $f \in D(A_3)$  (assuming the product makes sense)?

A positive answer to this question has been provided with extra assumptions on  $A_1$ ,  $A_2$  or (and) on the space  $X$ , for instance the following:

- \*  $A_1$  and  $A_2$  are continuous on  $X$ .
- \*  $-A_1$  is the generator of a linear contraction semigroup and  $A_2$  is continuous on  $X$ .

\*  $X$  is a Hilbert space and  $A_1$ ,  $A_2$ ,  $A_1 + A_2$  are single-valued maximal monotone operators (see Brézis-Pazy [2] or Brézis [1]).

\*  $X$  is a Hilbert space and  $A_1$ ,  $A_2$  are the subdifferentials of lower semi-continuous convex functions from  $X$  into  $]-\infty, \infty]$  (see Masuda-Kato [7]).

Other results are also mentioned in Kato [6]. It is interesting to notice that all the results above are (more or less easy) consequences of the nonlinear version

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of Chernoff's lemma (see [3]) given by Brézis-Pazy in [2] which says: given  $(U(t))_{t \geq 0}$ , a family of contractions from a closed convex subset  $C$  of  $X$  into itself, if there exists  $A_3$  m-accretive such that  $\overline{D(A_3)} = C$  and

$$\forall f \in C, \forall \lambda > 0, \lim_{t \rightarrow 0^+} \left[ I + \frac{\lambda}{t} (I - U(t)) \right]^{-1} f = (I + \lambda A_3)^{-1} f,$$

then

$$\forall f \in C, \forall t \in [0, \infty], \lim_{n \rightarrow \infty} \left[ U\left(\frac{t}{n}\right) \right]^n f = S_3(t)f.$$

The purpose of this paper is to give a counterexample showing that the question above has a negative answer in that general setting. Moreover we exhibit here two linear m-accretive operators  $A_1, A_2$  whose sum  $A_3 = A_1 + A_2$  is also m-accretive and for which (1) fails for some  $f \in \overline{D(A_3)}$  as well as

$$\forall t \in [0, \infty], \lim_{n \rightarrow \infty} \left[ \left( I + \frac{t}{n} A_1 \right)^{-1} \left( I + \frac{t}{n} A_2 \right)^{-1} \right]^n f = S_3(t)f.$$

To understand this counterexample with respect to Trotter's result, it is necessary to remember that an operator  $A$  on a Banach space  $X$  is said to be m-accretive if, for any  $\lambda > 0$ ,  $(I + \lambda A)^{-1}$  is a nonexpansive mapping defined on the whole space  $X$  (see e.g. [2] for more details). Consequently, by the well-known Hille-Yosida theorem, if  $A$  is a linear m-accretive operator,  $-A$  is the (infinitesimal) generator of a strongly continuous semigroup of contractions if and only if its domain  $D(A)$  is dense. Obviously this property fails in our examples below. Therefore, if these operators generate semigroups in the "nonlinear sense" (see Crandall-Liggett [5]), that is

$$(2) \quad \forall f \in \overline{D(A)}, \forall t \in [0, \infty], S(t)f = \lim_{n \rightarrow \infty} \left( I + \frac{t}{n} A \right)^{-n} f,$$

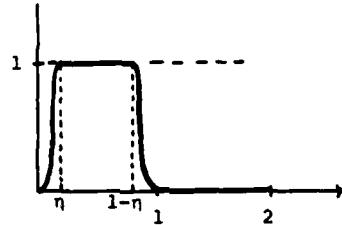
they are not strong generators of these semigroups.

Let  $C_b(\mathbb{R})$  (resp.  $C(K)$ ) denote the Banach space of the bounded continuous functions on  $\mathbb{R}$  (resp. on the compact set  $K$  of  $\mathbb{R}$ ) with the norm

$$\forall u \in C_b(\mathbb{R}), \quad \|u\| = \sup_{x \in \mathbb{R}} |u(x)|$$

$$(\text{resp. } \forall u \in C(K), \quad \|u\| = \sup_{x \in K} |u(x)|).$$

Let  $\rho \in C^\infty(\mathbb{R})$  be a periodic function with period 2 whose graph on  $[0,2]$  is:



On  $C_b(\mathbb{R})$ , we define the following operators (the derivative is taken in the sense of distributions).

$$(i) \quad D(A_1) = \{u \in C_b(\mathbb{R}); \quad \rho x^3 u' \in C_b(\mathbb{R})\}$$

$$A_1 u = \rho x^3 u' .$$

$$(ii) \quad D(A_2) = \{u \in C_b(\mathbb{R}); \quad (1 - \rho)x^3 u' \in C_b(\mathbb{R})\}$$

$$A_2 u = (1 - \rho)x^3 u' .$$

$$(iii) \quad D(A_3) = \{u \in C_b(\mathbb{R}); \quad x^3 u' \in C_b(\mathbb{R})\}$$

$$A_3 u = x^3 u' .$$

For any compact set  $K$  of  $\mathbb{R}$ , symmetric with respect to 0, we define on  $C(K)$ :

$$\forall i = 1, 2, 3, \quad D(A_i^K) = \{u \in C(K); \quad a_i x^3 u' \in C(K)\}$$

$$A_i^K u = a_i x^3 u' ,$$

where  $a_1 = \rho|_K$ ,  $a_2 = (1 - \rho)|_K$ ,  $a_3 = 1_K$ . Here the derivative is taken in  $D'(K)$  and " $a_i x^3 u' \in C(K)$ " means that  $a_i x^3 u'$  is continuous on  $K$  and can be continuously extended to  $\mathbb{R}$ .

PROPOSITION 1.

(i) For  $i = 1, 2, 3$ ,  $-A_i^K$  is the generator of a strongly continuous contraction semigroup  $S_i^K$  on  $C(K)$  and  $A_1^K + A_2^K = A_3^K$ .

(ii) For  $i = 1, 2, 3$ ,  $A_i$  is m-accretive on  $C_b(\mathbb{R})$  and  $A_1 + A_2 = A_3$ .

(iii) For  $i = 1, 2, 3$ ,

$$\forall f \in C_b(\mathbb{R}), \forall \lambda > 0, [(I + \lambda A_i)^{-1} f]_K = (I + \lambda A_i^K)^{-1} (f|_K).$$

(iv) If  $S_i(t) : \overline{D(A_i)} \rightarrow \overline{D(A_i)}$  is defined by

$$\forall f \in \overline{D(A_i)}, \forall t \geq 0, S_i(t)f = \lim_{n \rightarrow \infty} [I + \frac{t}{n} A_i]^{-n} f,$$

then:

$$\forall f \in \overline{D(A_i)}, \forall t \geq 0, [S_i(t)f]_K = S_i^K(t)(f|_K).$$

Remark 1. If  $u \in D(A_3), x^3 u'$  is bounded. Hence  $\lim_{x \rightarrow \infty} u(x)$  and  $\lim_{x \rightarrow -\infty} u(x)$  exist.

Therefore  $D(A_3)$  is not dense in  $C_b(\mathbb{R})$ .

Note also that, if  $x_n, y_n \in [2n + n, 2n + 1 - n]$  and if  $u \in D(A_1)$ , then:

$$|u(x_n) - u(y_n)| \leq \frac{1}{2} \|px^3 u'\| \left[ \frac{1}{x_n^2} + \frac{1}{y_n^2} \right].$$

This also proves that  $D(A_1)$  is not dense in  $C_b(\mathbb{R})$ .

PROPOSITION 2.

(i)  $S_1(t)$  and  $S_2(t)$  leave  $\overline{D(A_3)}$  invariant and for all  $f \in \overline{D(A_3)}$  and all  $t \in [0, \infty)$ ,  $[S_1(\frac{t}{n}) S_2(\frac{t}{n})]^n f$  converges to  $S_3(t)f$  uniformly on compact subsets of  $\mathbb{R}$ .

(ii) For all  $f \in C_b(\mathbb{R})$  and all  $t > 0$ ,  $[(I + \frac{t}{n} A_1)^{-1} (I + \frac{t}{n} A_2)^{-1}]^n f$  converges to  $S_3(t)f$  uniformly on compact subsets of  $\mathbb{R}$ .

But:

(iii) For any  $f \in C_b(\mathbb{R})$  with compact support and  $f \neq 0$ , there exists  $t \in (0, \infty)$  such that  $[S_1(\frac{t}{n}) S_2(\frac{t}{n})]^n f$  does not converge in  $C_b(\mathbb{R})$ .

For all  $t \in [0, \infty)$ , there exists  $f \in C_b(\mathbb{R})$  such that  $[S_1(\frac{t}{n}) S_2(\frac{t}{n})]^n f$  does not converge in  $C_b(\mathbb{R})$ .

(iv) For any  $f \in C_b(\mathbb{R})$  with compact support and  $f \neq 0$ , there exists  $t$  such that  $\left[ (I + \frac{t}{n} A_1)^{-1} (I + \frac{t}{n} A_2)^{-1} \right]^n f$  does not converge in  $C_b(\mathbb{R})$ .

Proof of Proposition 1.

The equalities  $A_1^K + A_2^K = A_3^K$ ,  $A_1 + A_2 = A_3$  follow directly from the definition.

For each  $i = 1, 2, 3$ , the proposition is a consequence of the following lemma.

Lemma. Let  $\alpha$  be a nonnegative function of  $C^{\infty}(\mathbb{R}) \cap C_b(\mathbb{R})$ . Let  $A$  (resp.  $A^K$ ) be defined on  $C_b(\mathbb{R})$  (resp.  $C(K)$ ) by

$$D(A) = \{u \in C_b(\mathbb{R}); \alpha x^3 u' \in C_b(\mathbb{R})\}, \quad Au = \alpha x^3 u'$$

$$(\text{resp. } D(A^K) = \{u \in C(K); \alpha x^3 u' \in C(K)\}, \quad A^K u = \alpha x^3 u') .$$

Then:

(i)  $-A^K$  is the generator of the strongly continuous semigroup of contractions  $S^K(t)$  on  $C(K)$  defined by

$$(3) \quad \forall f \in C(K), \quad S^K(t)f(x) = f(X(t,x)) ,$$

where  $X(\cdot, x)$  is the solution of

$$(4) \quad \frac{d}{dt} X(t,x) = -\alpha(X(t,x))X^3(t,x), \quad X(0,x) = x .$$

Moreover, for all  $\lambda > 0$

$$(5) \quad \forall f \in C(K), \quad \forall x \in K, \quad (I + \lambda A^K)^{-1}f(x) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{t}{\lambda}} f(X(t,x)) dt .$$

(ii)  $A$  is m-accretive on  $C_b(\mathbb{R})$  and

$$\forall f \in C_b(\mathbb{R}), \quad \forall x \in \mathbb{R}, \quad (I + \lambda A)^{-1}f(x) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{t}{\lambda}} f(X(t,x)) dt ,$$

$$\forall f \in \overline{D(A)}, \quad \forall x \in \mathbb{R}, \quad S(t)f(x) = f(X(t,x)) ,$$

where  $S(t)$  is defined by (2).

Proof of the Lemma.

The proof of (i) is similar to the proof of Theorem (1.1) in [8].

Since  $K$  is symmetric and since  $[x \mapsto -\alpha(x)x^3]$  is Lipschitz continuous on  $K$  and has the same sign as  $-x$ , (4) has a unique solution which stays in  $K$  for  $x \in K$

and satisfies

$$\forall t \geq 0 \quad |x(t,x)| \leq |x|$$

$(t,x) \in [0,\infty[ \times K \rightarrow X(t,x)$  is continuous.

It follows that (3) defines a strongly continuous semigroup of contractions  $S^K(t)$  on  $C(K)$  whose generator  $L$  is given by

$$Lu(x) = \lim_{\substack{+ \\ t \rightarrow 0}} \frac{u(X(t,x)) - u(x)}{t},$$

when the limit exists uniformly in  $x \in K$ . Proceeding as in [8], we prove that  $L$  is the closure of its restriction  $L_0$  to  $C^1(K)$ . Indeed let  $L$  denote the Lipschitz continuous functions on  $K$ . Then, if  $u \in D(L) \cap L$

$$Lu(x) = -\alpha(x)x^3u'(x),$$

and  $[u, Lu]$  is the limit in  $C(K) \times C(K)$  of some  $[u_n, L_0 u_n]$  with  $u_n \in C^1(K)$ . This proves that  $\overline{L_0}$  contains the restriction of  $L$  to  $D(L) \cap L$ . But one can show that  $L$  is the closure of this restriction by using the fact that  $S(t)$  leaves  $D(L) \cap L$  invariant.

Now let us show  $-\overline{L_0} = A^K$ . If  $[u_n, \alpha x^3 u'_n] \in -L_0$  converges to  $[u, v]$  in  $C(K) \times C(K)$ , then  $\alpha x^3 u'_n$  converges to  $\alpha x^3 u'$  in the sense of distributions; hence  $\alpha x^3 u' = v \in C(K)$  which proves  $-\overline{L_0} \subset A^K$ .

For the converse, as  $I - \overline{L_0}$  is onto on  $C(K)$ , it is sufficient to remark that  $I + A^K$  is one-one, that is:

$$(6) \quad (u \in C(K), \quad u + \alpha x^3 u' = 0 \text{ in } D^*(K)) \implies (u = 0 \text{ on } K).$$

This achieves the proof of (i), the property (5) being well-known.

To prove that  $A$  is m-accretive, let us consider for  $f \in C_b(\mathbb{R})$  and  $\lambda > 0$ :

$$(7) \quad u_\lambda(x) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{t}{\lambda}} f(X(t,x)) dt.$$

For any  $K$  as above, we have

$$\forall x \in K, \quad u_\lambda(x) = (I + \lambda A^K)^{-1}(f|_K)(x).$$

As  $K$  is arbitrary, this proves that  $u_\lambda$  and  $\alpha x^3 u'_\lambda$  are continuous on  $\mathbb{R}$  and verify

$$u_\lambda + \lambda \alpha x^3 u'_\lambda = f \text{ in } D'(\mathbb{R}).$$

Since  $\|u_\lambda\| \leq \|f\|$  by definition,  $u_\lambda$  and  $\alpha x^3 u'_\lambda \in C_b(\mathbb{R})$ . Hence  $u_\lambda \in D(A)$  and  $u_\lambda + \lambda A u_\lambda = f$ .

This proves that  $A$  is an extension of an  $m$ -accretive operator. Since  $I + A$  is one-one (see (6)),  $A$  is  $m$ -accretive.

The relations (5) and (7) give

$$\forall f \in C_b(\mathbb{R}), [(I + \lambda A)^{-1} f]_K = (I + \lambda A^K)^{-1} (f|_K).$$

Hence, by the definition (2):

$$\forall f \in \overline{D(A)}, S(t)f|_K = \lim_{n \rightarrow \infty} [I + \frac{t}{n} A^K]^{-n} (f|_K) = S^K(t)(f|_K).$$

(The last equality is well-known for the linear generators.) Finally

$$\forall f \in \overline{D(A)} \quad S(t)f(x) = f(X(t,x)).$$

Remark 2. If  $\alpha \equiv 1$  (i.e.  $A = A_3$ ), we obtain that

$$X(t,x) = \frac{\operatorname{sgn} x}{\sqrt{2t + \frac{1}{x^2}}}.$$

Then,  $S(t)f(x) = f(X(t,x))$  defines a semigroup of contractions on  $C_b(\mathbb{R})$ , but one can directly verify that  $t \mapsto S(t)f$  is continuous at 0 if and only if  $f \in C(\bar{\mathbb{R}}) = \{g \in C_b(\mathbb{R}); \lim_{x \rightarrow \infty} g(x) \text{ and } \lim_{x \rightarrow -\infty} g(x) \text{ exist}\}$ . Since  $\tilde{S}(t)$  leaves  $C(\bar{\mathbb{R}})$  invariant and since  $\overline{D(A_3)} \subset C(\bar{\mathbb{R}})$  by the remark 1,  $S_3(t)$  is exactly the restriction of  $\tilde{S}(t)$  to  $C(\bar{\mathbb{R}})$  and  $C(\bar{\mathbb{R}}) = \overline{D(A_3)}$ .

#### Proof of Proposition 2.

Observe that, by the definition of  $\rho$ , for  $i = 1, 2$ :

$$(8) \quad \begin{cases} \forall x > 0, x - 1 - \eta \leq X_i(t,x) \leq x \\ \forall x < 0, x \leq X_i(t,x) \leq x + 1 + \eta. \end{cases}$$

$(X_i, i = 1, 2)$ , is the solution of (4) with  $\alpha_1 = 1, \alpha_2 = 1 - \beta$ . Therefore,

(which is the set  $\{g \in C_b(\mathbb{R}) : \lim_{x \rightarrow +\infty} g(x) \text{ and } \lim_{x \rightarrow -\infty} g(x) \text{ exist by Remark 1}\}$ )

invariant under  $S_1(t)$  and  $S_2(t)$ . Hence  $[S_1(\frac{t}{n})S_2(\frac{t}{n})]^n f$  is defined for all  $f \in C_b(\mathbb{R})$ .

Then, using (i), (iii) and (iv) in proposition 1, parts (i) and (ii) are consequences of Trotter and Chernoff's results (see [10], [3]).

Now by (8), if  $f \in C_b(\mathbb{R})$  has compact support in  $[-R, R]$ ,  $S_1(t)f$  and  $S_2(t)f$  also have compact support in  $[-R - 1 - n, R + 1 + n]$  for any  $t > 0$  and so do

$(I + tA_1)^{-1}f$  and  $(I + tA_2)^{-1}f$  by (ii) in the lemma.

So let  $f \in C_b(\mathbb{R})$  have compact support and assume that  $[S_1(\frac{t}{n})S_2(\frac{t}{n})]^n f$  or  $[(I + \frac{t}{n}A_1)^{-1}(I + \frac{t}{n}A_2)^{-1}]^n f$  converge uniformly on  $\mathbb{R}$ . The limit is necessarily  $S_3(t)f$  which is given by:

$$\forall t > 0, \forall x \neq 0, S_3(t)f(x) = f\left(\frac{\operatorname{sgn} x}{\sqrt{2t + \frac{1}{x^2}}}\right).$$

Then we have

$$0 = S_3(t)f(+\infty) = f\left(\frac{1}{\sqrt{2t}}\right), \quad 0 = S_3(t)f(-\infty) = f\left(\frac{-1}{\sqrt{2t}}\right).$$

If  $f \neq 0$ , this is false for some  $t \in (0, \infty)$ .

For the last statement of (iii), given  $t > 0$ , let  $f \in C_b(\mathbb{R})$  have compact support and  $f = 1$  on  $[-\frac{1}{\sqrt{2t}}, \frac{1}{\sqrt{2t}}]$ . Then

$$S_3(t)f \equiv 1.$$

Clearly  $[S_1(\frac{t}{n})S_2(\frac{t}{n})]^n f$ , which has compact support, cannot converge uniformly to 1.

Remark 3. If  $\hat{C}(\mathbb{R})$  denotes the continuous functions on  $\mathbb{R}$  which vanish at  $+\infty$ ,

let  $\hat{A}_i = A_i \cap \hat{C}(\mathbb{R}) \times \hat{C}(\mathbb{R})$ . Then we can show that  $-\hat{A}_1, -\hat{A}_2$  are the (strong) generators of continuous semigroups of contractions  $\hat{S}_1(t), \hat{S}_2(t)$ . The same remarks as above prove that  $[\hat{S}_1(\frac{t}{n})\hat{S}_2(\frac{t}{n})]^n f$  do not always converge in  $\hat{C}(\mathbb{R})$ . (Obviously

$-\hat{A}_3$  does not generate any semigroup even in the nonlinear sense.) Trotter also noted in [10] that the convergence of this product may fail for the sum of two generators.

Let us finally recall the example given by Pitt [9] showing that, if  $-\hat{A}_1$ ,  $-\hat{A}_2$  are two generators, the above product may converge even if  $D(A_1) \cap D(A_2) = \{0\}$ . See also Chernoff [4] for more pathological cases.

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